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# Experimental testing of a stochastic sediment transport model

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Wash-off;
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**Summary** A stochastic model of sediment transport by rainfall-runoff was tested with a simple laboratory experiment. Although the conceptual basis of the model has been previously published [Lisle, I.G., Rose, C.W., Hogarth, W.L., Hairsine, P.B., Sander, G.C., Parlange, J.-Y., 1998. Stochastic sediment transport in soil erosion. Journal of Hydrology 204, 217–230] and its mechanistic underpinnings were convincingly theorized, it had not been corroborated with measurements. Small-scale flume experiments, ~0.8 m long, with simulated rainfall were used to imitate ''wash-off'' of sediment (0.225 mm silica sand) from an impervious surface. Fitting two parameters (ejection and deposition rates) to minimize least squares error resulted in good agreement between stochastic model and measurements, $R^2 \sim 0.9$. However, the fitted parameter values differed from values that would be expected following Lisle et al.'s (1998) physical explanation of the ejection and deposition rates. While some of the discrepancy may be attributable to particle interactions, the conceptualization of deposition as a quiescent settling process in mechanistic erosion models may need to be reevaluated.


## Introduction

We tested a stochastic sediment transport model (Lisle et al., 1998) against small-scale, laboratory observations of time dependent particle loss on a rough, impervious surface where particle ejection is primarily driven by rainfall, not overland shear. Most previous experiments of rainfall-driven, interrill erosion either only quantify steady-state loss (Chaplot and Le Bissonnais, 2000; Jayawardena and Bhuiyan, 1999; Zartl et al., 2001) or involve many simultaneously occurring dynamic processes that make it difficult to isolate fundamental physical mechanisms (Proffitt et al., 1991).

While capable of elucidating fundamental processes, the model is not without limitations; Lisle et al. (1998) note that

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their stochastic model is most useful to predict movement of particles present in a finite quantity with limited interactions with other particles or surrounding media. Experiments strictly on soils do not generally fulfill these conditions. However, an ideal scenario for testing the Lisle et al. (1998) model is ''wash-off'' from an impervious surface with a finite supply of sediment. For this experimental condition, the critical assumption within the Lisle et al. (1998) model — that the probabilistic position of a single particle can represent the aggregate movement of multiple particles — is most likely attained.

Additionally, while an idealization of soil erosion, the Lisle et al. (1998) model may be directly applicable to modeling particulate transport processes in urban catchments. With the exception of Shaw et al.'s (2006) model that followed the deterministic approach of the Hairsine and Rose (1991) soil erosion model, previous wash-off models have primarily been empirical, were fitted only by calibration, and gave little insight into underlying physical wash-off processes (Akan, 1988; Deletic et al., 1997). This paper has two objectives: (1) to test the model proposed by Lisle et al. (1998) against experimental wash-off data and (2) to consider the physical basis of the model parameters.

## Theory

Following Lisle et al. (1998), particle movement is assumed to be a Markov process with particles alternating between a state of rest or a state of motion, with the probability of being in one state or the other signified by $q$ (at rest) and $p$ (in motion). The spatial distribution of particles in motion or at rest across a one-dimensional surface is described by the ''Kolmogorov—Feller'' equations:

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} = -kp + hq \quad (1a)$$

$$\frac{\partial q}{\partial t} = kp - hq \quad (1b)$$

where $u$ is the bulk flow velocity of the overland flow, $k$ is a settling rate constant, and $h$ is an ejection rate constant (Fig. 1). For the initial conditions $p(x,0) = \delta(x)$ and $q(x,0) = 0$, the system of Eqs. (1a) and (1b) is solved as:

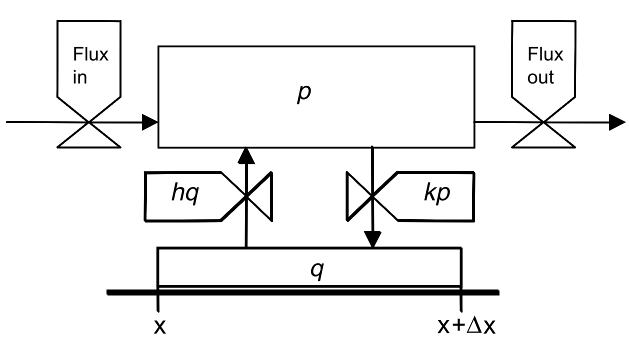


**Figure 1** Schematic of rainfall-driven particle movement in shallow flow described by Eqs. (1a) and (1b). $p$ is the probability of a particle being in motion while $q$ is the probability a particle is at rest on the rough surface. $k$ scales the rate of ejection of particles at rest while $h$ scales the rate of deposition of particles in motion. Particles in motion move laterally with a velocity $u$.

$$p_\delta(x,t) = H(\tau)H(\xi)\frac{h}{u}e^{-\xi}e^{-\tau}\sqrt{\frac{\xi}{\tau}}I_1\left[2\sqrt{(\xi\tau)}\right] + \frac{H(\xi)}{u}e^{-\xi}\delta(\tau) \quad (2)$$

where $\xi = \frac{kx}{u}$, $\tau = h(t - \frac{x}{u})$, $H()$ is the Heaviside step function, $\delta()$ is the Dirac delta function, and $I_1$ is a Modified Bessel function of the 1st order (Lisle et al., 1998).

For an initial condition of $p(L > \hat{x} > 0, 0) = C_0$, analogous to complete particle coverage of a surface length $L$, a convolution integral may be used:

$$p(x,t) = \int_{-\infty}^{\infty} p(\hat{x},0)p_\delta(x-\hat{x},t)d\hat{x} \quad (3)$$

As implied by the use of the convolution, we assume our experimental system will respond linearly due to the relatively low spatial density of particles on the flume surface (e.g. even with particle coverage of the entire flume surface, we assume there is not significantly greater interaction between particles than when we assume a pulse is applied).

The arrival time density ($r$) is given as

$$r(x,t) = up(x,t) \quad (4)$$

where this represents a rate of mass loss appropriate for plotting as a breakthrough curve at a fixed position of $x$.

Stochastically, the function $r(x = L, t)$ establishes the probability of a single particle being located at a position $L$ at a time $t$. When considered in terms of the movement of many particles, $r(x = L, t)$ is the normalized particle loss at a position $L$ at time $t$.

## Methods

The experimental set-up and collection methods are only briefly summarized here and a more complete description is presented in Shaw et al. (2006). Rainfall and upslope flow were independently applied to an 80 cm long, 10.5 cm wide (width = $W$) stainless steel flume with a 4% slope (Fig. 2). A rough, flume bed was cut from a sheet of prismatic, polycarbonate diffuser used for recessed fluorescent light fixtures. A small Plexiglas stilling chamber with an overflow weir was used to control upslope flow. Three meters above the flume, rainfall was generated by four hypodermic needles that oscillated along two orthogonal tracks attached to the ceiling of the Soil and Water Lab at the Cornell University Department of Biological and Environmental Engineering. Using the flour pellet method (Laws and Parsons, 1943), the average raindrop radius was 0.082 cm at a rainfall rate of 0.13 cm min$^{-1}$. An empirical relationship between flow and velocity was developed from five velocity measurements made over a range of overland flow rates applied only as upslope inflow. Flow velocity was determined by measuring the average time for a pulse of dye (FD&C red dye No. 40), injected into the flow stream with a pipette, to travel 40 cm; the measurement was made over the middle section of the flume to avoid end-effects. Velocities measurements were made in triplicate and the coefficient of variation was $<\sim 5\%$.

We applied 225 μm silica sand particles to the flume surface using two different initial conditions: a 2 cm band intended to represent a delta pulse (Runs 1 and 2) and complete coverage of 40 cm of the flume bed (Run 3). Par-

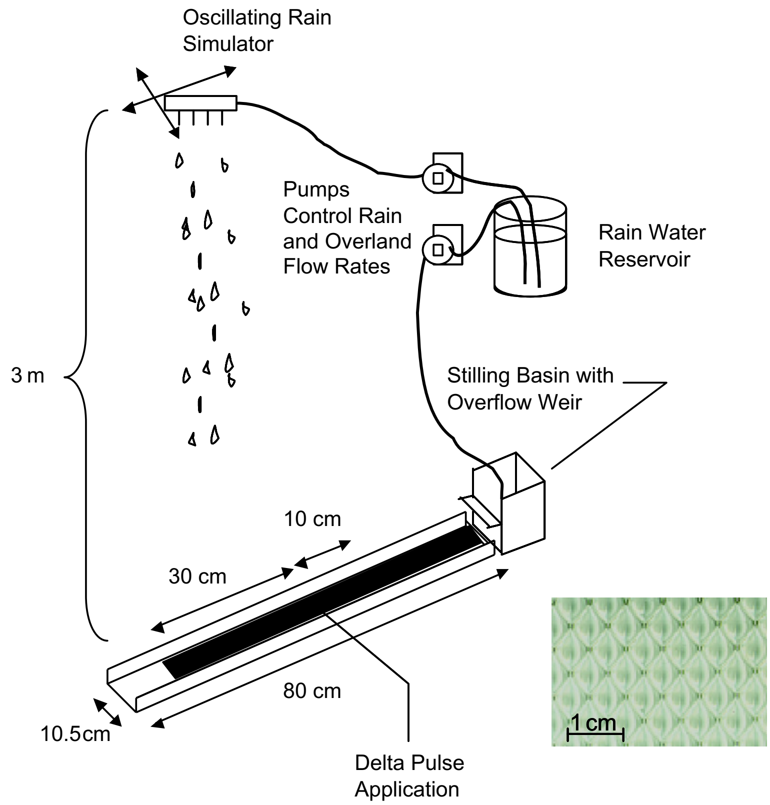


**Figure 2** Schematic of experimental set-up and apparatus. Overland flow can be generated independently of rainfall by spilling water from the stilling basin. Inset is a scale image of the prismatic, polycarbonate diffuser plate used as the roughness surface in the experiments. Indentations are approximately 1 mm deep.

ticle spatial densities below 0.03 g cm$^{-2}$ were used in order to minimize interactions between particles (Shaw et al., 2006). All runs were completed in duplicate. Flow off the end of the flume was funneled through a medium paper filter (Fisher Scientific). Each run was typically 10 min long and filters were changed every 45 s.

## Results

The model (Eqs. (4) and (2)) was fit to the experimentally observed mass loss for a sediment pulse applied at the top of the flume (Runs 1 and 2 as shown in Fig. 3a and b). This scenario duplicates the conditions discussed in Lisle et al. (1998). In the model, the $x$-origin was taken at the center of the 2 cm pulse. ''Fitting'' entailed adjusting $h$ and $k$ in Eq. (2) to minimize the least squares error between simulated and observed (recall $u$ was known from direct measurement). The model was also tested on conditions more closely approaching a real-world scenario in which particles completely cover a surface. The model (Eqs. (2)–(4)) was fit to the wash-off from this initial condition (i.e. Run 3 as shown in Fig. 4), again minimizing least squares error between observed and simulated. The convolution (Eq. (3)) was solved numerically with a spatial increment of 2 cm.

We find that the Lisle et al. (1998) model can be used to suitably replicate the observed data in all cases ($R^2 > 0.85$). Of particular note, despite not including an explicit dispersion term, the stochastic model inherently results in a dispersed pulse. Table 1 summarizes the fitted $h$ and $k$ values as well as the $R^2$ values for each run. As would be expected, $h$, the ''ejection rate'', increases with increasing $P$ while $k$, the ''deposition rate'', remains constant. Surprisingly though, for Run 3, $h$ is not the same value as for Run 2 where the same $P$ is used, possibly indicating that more interaction between particles occurs than we assumed or the model accounts for. We will address this point further in the Discussion.

As an illustration of model sensitivity to parameters, Fig. 5 shows that when $h$ is shifted above and below the best-fit value by 10%, the timing and magnitude of the peak also change by approximately 10%. Thus, any *a priori* estimate of $h$ will require accurate estimation of underlying physical parameters, an issue to be discussed next.

## Discussion

Ideally, one would like to be able to select appropriate $h$ and $k$ values without model fitting. While stochastically, $h$ and $k$ parameterize the distribution of durations of rest and motion, respectively, Lisle et al. (1998) also attributed a physical meaning to $h$ and $k$:

$$h = \frac{A_0}{V}P \tag{5}$$

$$k = \frac{v}{l} \tag{6}$$

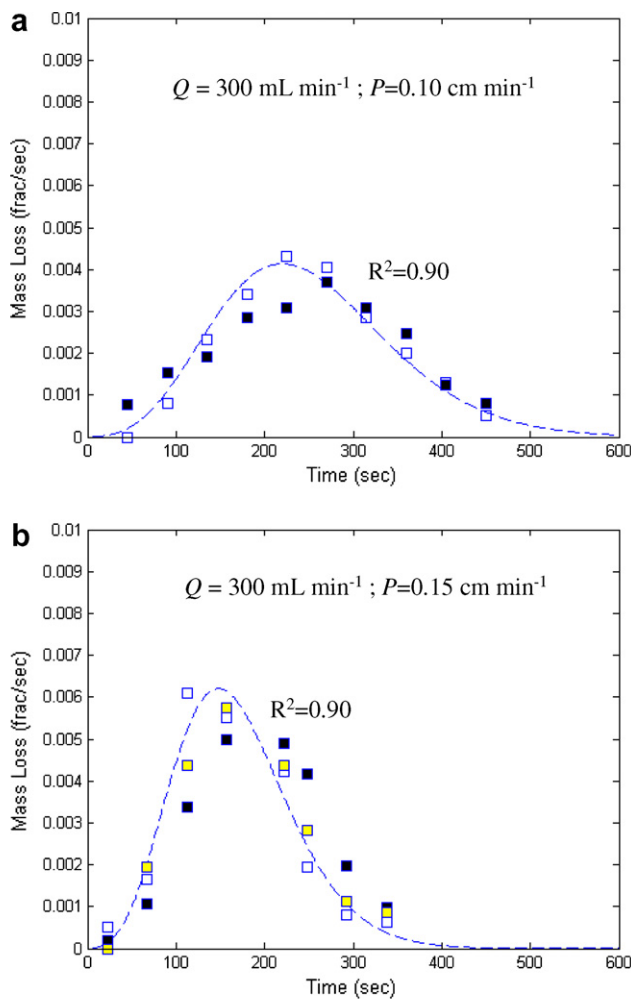


**Figure 3** Lisle et al. (1998) model fit to experimental Run 1 (b) and Run 2 (a). Differing symbols indicate replicate trials. The dashed line indicates the model. The experimental runs used an initial 2 cm wide pulse of sand to replicate the delta function initial condition.

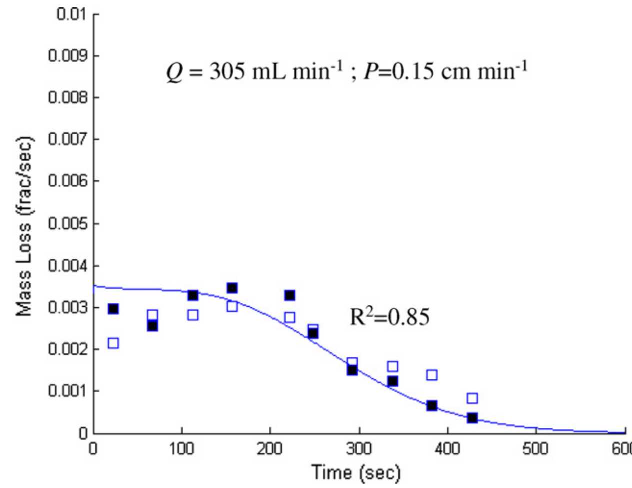


**Figure 4** Model fit to experimental Run 3. Filled and open symbols indicate two duplicate trials. The solid line indicates the model. For this run, the surface initially was covered with sand grains from 0 to 0.4 m.

where $A_0$ is the impact area of a rain drop ($cm^2$), $V$ is the rain drop volume ($cm^3$) estimated from the measured drop radius assuming the drops are roughly spherical, $P$ is the rainfall rate ($cm\,min^{-1}$), $v$ is the settling velocity ($cm\,min^{-1}$) and $l$ is the flow depth (cm). Based on the formulation of Eqs. (5) and (6), the magnitude of $h$ reflects the fraction of surface area disturbed by rain drops per unit time, and the magnitude of $k$ reflects the rate of time for a particle to settle over the flow depth, inherently implying a vertical settling process.

The accuracy of estimates of $h$ and $k$ from Eqs. (5) and (6) is partially dependent on the estimates of the underlying physical quantities. Water depth ($l$) can be estimated using continuity ($Q/[u\ W] = l$ where with $Q = 5.36$ mL/s (the average total flow for Run 1, 322 mL/min), $u = 8.3$ cm/sec, and $W = 10.5$ cm, $l$ is 0.062 cm). While this is the hydraulic flow depth (i.e. the depth of the flow profile that does not include the quiescent region within the roughness elements), a proper settling depth ($l$) would require subtraction of the particle diameter from the total depth. However, given the similar lengths of the roughness elements and the particles, the hydraulic flow depth was considered a reasonable proxy for the actual distance across which the particle would settle, i.e., $l \approx 0.062$ cm. An average settling velocity was determined experimentally (grains were hand-timed falling 40.0 cm in a 1 L graduated cylinder); based on the average of three replicates, $v \approx 170\ cm\,min^{-1}$.

But, unlike $l$ and $v$, $A_0$ cannot be determined easily from either direct or indirect measurements, and a best estimates can only be drawn from highly controlled experiments on drop impact (Macklin and Metaxas, 1976; Prosperitti and Oguz, 1993) as well as inferences from experiments on aggregate mass loss from a flume (Shaw et al., 2006). A simple way to generalize $A_0$ from experiments using various drop sizes is to consider the related quantity of the ratio between drop impact area radius and the drop radius. Previous investigators have suggested drop impact-to-drop radius values ranging from approximately three (Macklin and Metaxas, 1976) to seven (Prosperitti and Oguz, 1993; Shaw et al., 2006).

Using the above estimates of the underlying physical quantities, $h$ and $k$ can be calculated as follows from Eqs. (5) and (6). Assuming the drop impact-to- drop radius ratio needed to determine $A_0$ is approximately five (taking the mean of the available estimates) and using the 0.082 cm drop radius to calculate $V$, $h$ should be $\sim 0.39\ s^{-1}$ and $\sim 0.59\ s^{-1}$ for $P$ of 0.10 and 0.15 $cm\,min^{-1}$, respectively. Using Eq. (6) with $l = 0.062$ cm and $v = 170\ cm\,min^{-1}$, $k$ should be $46\ s^{-1}$. Thus, we find that the model fitted parameters $h$ and $k$ (Table 1) are much smaller than the values that would be most expected given the physical interpretation of the parameters in Eqs. (5) and (6).

To add some additional insight into the relationship between $h$ and $k$, we note that Lisle et al. (1998) also related $h$ and $k$ to effective particle velocity, $u_{eff}$, and particle dispersion, $D$, for a long-time solution to Eqs. (1a) and (1b):

$$u_{eff} = \frac{h}{h+k}u \tag{7}$$

$$D \sim \frac{hk}{(h+k)^3}u^2 \tag{8}$$

**Table 1** Summary of model parameters for each run

| Run | $Q$ (mL/min) | $P$ (cm/min) | Initial $M_0$ (g cm$^{-2}$) | $u$ (cm s$^{-1}$) | $h$ (s$^{-1}$) | $k$ (s$^{-1}$) | $R^2$ |
|---|---|---|---|---|---|---|---|
| 1 | 300 | 0.10 | 0.03 | 8.3 | 0.050 | 3.2 | 0.90 |
| 2 | 300 | 0.15 | 0.03 | 8.6 | 0.070 | 3.2 | 0.90 |
| 3 | 305 | 0.15 | 0.02 | 8.6 | 0.050 | 3.2 | 0.85 |

Note, in Runs 1 and 2, particulate was applied in a 2 cm wide strip. In Run 3, particulate was applied over 40 cm of the flume surface.

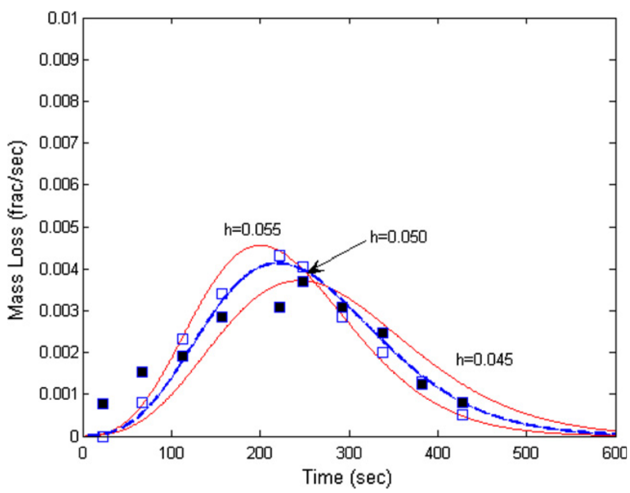


**Figure 5** Sensitivity of Run 1 to a 10% change in $h$.

The long-time solution resembles the familiar advective-dispersion equation, thereby translating somewhat abstract $h$ and $k$ values into the more familiar quantities of dispersion and particle velocity. Any properly proportioned ratio of $h$ and $k$ can result in a reasonable $u_{\mathrm{eff}}$, but including $D$ constrains $h$ and $k$ for a given set of experimental conditions to one unique pair, a fact relevant to the following discussion.

The failure of Eqs. (5) and (6) to suitably predict $h$ and $k$ values suggests at least two explanations, either $A_0$ and $v$ are poorly estimated or the stochastic behavior of a single particle does not accurately represent the aggregate movement of many particles. First, given that our estimate of $A_0$ is admittedly based on limited information, $A_0$ could reasonably be lower. But — in following the ratios established by Eqs. (7) and (8) — we would need to also decrease $v$, thus implying that deposition was not well modeled by a quiescent settling process. Given that particle movement during erosion is infrequently conceptualized as a series of step-like movements (Sander et al., 2007), there has been little motivation to make direct observations of the deposition process in very thin flows. In the few cases when deposition is included as a separate process in a model, settling rates are typically assumed to be known and ejection rates are used to fit the model (Shaw et al., 2006; Sander et al., 1996). Therefore, there is some possibility that deposition in thin flows has been consistently incorrectly modeled as a quiescent settling process when, in reality, it may be more of a capture process dependent on factors other than quiescent settling velocity. Supporting this notion, Parsons and Stromberg (1998) found by experiment a power law relationship between particle diameter and travel distance different than would be expected assuming Eq. (6). Additionally, Nino et al. (2003), by direct visualization of particle movement in ~5 cm deep flow, did not even assume particle movement had occurred unless a particle traveled at least 100 particle diameters after ejection, a distance at least twice as long as would be expected given quiescent settling in their experiments. Thus, while evidence is limited, the deposition process may be much different than a simple settling process.

Alternatively, as a second possible reason Eqs. (5) and (6) fail to predict suitable $h$ and $k$ values, the model may be limited by the key assumption that the probabilistic position of a single particle can represent the aggregate movement of multiple particles. Particle collisions could effectively decrease the settling rate by keeping a particle aloft longer (thus decreasing the $k$ term) as well as decrease the effective drop impact area by damping impact energies and reducing particle ejection into the bulk flow (thus decreasing the $h$ term). The settling rate and impact area could be adjusted to act as effective parameters — differing from the direct physical measurement of isolated phenomenon — or an explicit dispersion term could be added to Eq. (1a).

## Conclusion

The stochastic erosion model by Lisle et al. (1998) was able to replicate experimental data when model parameter values were fitted. However, the model-fitted parameters did not agree with the physical interpretation of the parameters (Eqs. (5) and (6)), suggesting a need to account for particle interactions and, perhaps more importantly, to reassess the way in which deposition in thin flows is modeled.

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