IAHR AIRH

Journal of Hydraulic Research Vol. 46, No. 5 (2008), pp. 703–706
doi:10.3826/jhr.2008.3194

A note on Chow's description of the weak hydraulic jump

Note sur la description de Chow du ressaut hydraulique faible

WILLIAM HOGARTH, Pro Vice Chancellor, *Faculty of Science and Information Technology, The University of Newcastle, Callaghan, N.S.W., Australia, 2308. Tel.: 61 2 4921 5906; fax: 61 2 4921 7949; e-mail: Bill.Hogarth@newcastle.edu.au*

JEAN-YVES PARLANGE, Professor, *Department of Biological and Environmental Engineering, Cornell University, Ithaca, N.Y., U.S.A. 14583. Tel.: 1607 255 2476; fax: 1607 255 4080; e-mail: jp58@cornell.edu*

C.W. ROSE, Emeritus Professor, *Faculty of Environmental Sciences, Griffith University, Nathan, Qld, Australia, 4111. E-mail: C.Rose@griffith.edu.au*

GRAHAM SANDER, Reader, *Department of Civil and Building Engineering, Loughborough University, Leicestershire, UK. LE11 3TU. Tel.: 44 1509 223777; fax: 44 1509 223981; e-mail: g.sander@lboro.ac.uk*

MICHAEL TODD WALTER, Assistant Professor, *Department of Biological and Environmental Engineering, Cornell University, Ithaca, N.Y., U.S.A. 14583. Tel.: 1607 255 2488; fax: 1607 255 4080; e-mail: mtw5@cornell.edu*

MICHAEL FAIVRE WALTER, Professor, *Department of Biological and Environmental Engineering, Cornell University, Ithaca, N.Y., U.S.A. 14583. Tel.: 1607 255 2465; fax: 1607 255 4080; e-mail: mfw2@cornell.edu*

ABSTRACT
Standard analytical and empirical results of hydraulic jumps are reanalyzed. The assumption that momentum but not energy is conserved across a jump is abandoned. The limiting case when energy dissipation is maximum is explicitly considered, which could be useful when hydraulic jumps are used to dissipate energy. Chow's empirical results and some recent experimental observations are compared to the present predictions.

RÉSUMÉ
Les résultats analytiques et empiriques standard des ressauts hydrauliques sont ici reconsidérés. L'hypothèse selon laquelle c'est la quantité de mouvement et non pas l'énergie qui est conservée à travers un ressaut est abandonnée. Le cas limite de la dissipation d'énergie maximum est explicitement considéré, ce qui pourrait être utile quand des ressauts hydrauliques sont utilisés pour dissiper l'énergie. Les résultats empiriques de Chow et quelques observations expérimentales récentes sont comparés aux prévisions présentées.

Keywords: Chow, energy dissipation, hydraulic jump, momentum, open channel

1 Introduction

The theory of open-channel hydraulics, on which the design of hydraulic structures relies, owes much to Ven Te Chow (1914–1981). By an elegant mixture of theory, basically conservation of mass, momentum and energy, and empiricism, his results still form the basis of much of what we understand today. Of particular interest is the formation of hydraulic jumps, which have many practical applications (Chow, 1959) (see Fig. 1). One of them is its use as an energy dissipator. In the following Chow's description of hydraulic jumps is re-examined to gain some new theoretical understanding of experimental results and their use as energy dissipators.

2 Theoretical considerations

The standard steady state St. Venant equation can be written based on Darrigol (2002) as

$$\frac{dh}{dx} = gh^3(S_o - S_f)/(gh^3 - q^2) \tag{1}$$

where h is the flow depth, x the streamwise distance, S_o the bottom slope, q the discharge, g the acceleration of gravity and S_f the friction slope. The latter can be defined in terms of the normal depth h^* using a Chezy type relation as

$$S_f = S_o h^{*3}/h^3. \tag{2}$$

Revision received October 8, 2007/Open for discussion until April 30, 2009.

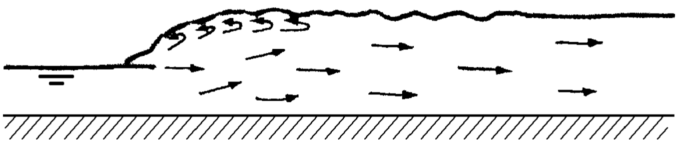

Figure 1 Illustration of weak jump as given by Chow (1959)

Equation (1) can be rewritten from conservation of momentum M as

$$(h - q^2/gh^2)\mathrm{d}h = hS_o\,\mathrm{d}x(1 - h^{*3}/h^3), \tag{3}$$

or from conservation of energy E as

$$(1 - q^2/gh^3)\mathrm{d}h = S_o\,\mathrm{d}x(1 - h^{*3}/h^3) \tag{4}$$

which are of course identical to Eq. (1) because of the steady state conditions ignoring correction factors (Liggett, 1994). Integration of Eq. (3) across the jump yields

$$\frac{h_1^2}{2} + \frac{q^2}{gh_1} - \frac{h_S^2}{2} - \frac{q^2}{gh_S} + S_o\int_0^L h\,\mathrm{d}x = M, \tag{5}$$

where h_1 is the value of h upstream of the shock and h_S at the end of it, and integration of Eq. (4) gives

$$\frac{q^2}{2gh_1^2} - \frac{q^2}{2gh_S^2} + h_1 - h_S + S_oL = E, \tag{6}$$

where L is the thickness of the shock (it is not taken necessarily as a discontinuity) and M and E are the momentum and energy loss due to the jump. From Eqs (3) and (4)

$$M = S_o\int_0^L \frac{h^{*3}}{h^2}dx \tag{7}$$

and

$$E = S_o\int_0^L \frac{h^{*3}}{h^3}dx. \tag{8}$$

In the standard treatment, Chow (1959) suggests that M can be neglected in Eq. (5), whereas E in Eq. (6) cannot. This clearly does not have to be the case, e.g. because of non negligible drag on a rough bed (Ead and Rajaratnam, 2002) or three-dimensional effects (Montes and Chanson, 1998).

To continue it is necessary to guess the shape of the jump. We have tried several "reasonable" shapes from a discontinuity to a parabola without affecting our conclusions. Thus we take here the simplest possible case with the jump being discontinuous at $x = 0$ and L. This does not mean that M and E in Eqs (7) and (8) are necessarily zero since h^* is then theoretically infinite within the jump, that is, taking $L = 0$ is only for mathematical convenience, in practice L is far from zero, as discussed below. Chow (1959) obtained his Eqs (15–18) when $G = F_1 = V_1/(gh_1)^{1/2}$ as the Froude number of the approach flow as

$$h_S/h_1 = 0.5[(1 + 8F_1^2)^{1/2} - 1] \tag{9}$$

as obtained from Eq. (5) with $M = L = 0$ with the Froude number also given by

$$F_1^2 = \frac{q^2}{gh_1^3}. \tag{10}$$

Then, Eq. (6) gives after elimination of F_1^2 using Eq. (9) (Chow, 1959)

$$\frac{E}{h_1} = \frac{(h_S/h_1 - 1)^3}{[4(h_S/h_1)]}. \tag{11}$$

Outside the jump Chow (1959) made the reasonable assumption that dissipation is negligible. Note that Hogarth *et al.* (2003) took $h^* = h_1$ at the upstream end of the jump, but it is easy to check that taking either h_1 or 0 has essentially no effect on the results; since the primary interest is discussing Chow's approach, his assumption is kept. Thus in Chow's case, Eq. (1) with $S_f = 0$ yields

$$S_ox = h - h_S + \frac{q^2}{2gh^2} - \frac{q^2}{2gh_S^2}. \tag{12}$$

If we do not assume, as Chow does, that $M = 0$ within the discontinuity at $x = 0$, some other condition must be imposed. It is clear that the standard approach, with $M = 0$, corresponds to the minimum energy dissipation. Here then, the other limiting case of maximum energy dissipation is considered which would be a useful approach in practice. Hopefully experimental observations will fall between those two limits, with laboratory experiments conducted on very smooth beds closer to the case of $M = 0$, and those with rough beds closer to "our case". The value of h_S such that E is maximum is obtained by differentiation of Eq. (6). In this case the jump has a tailwater depth $h = h_2$, at $x = 0$, writing h_2 to differentiate from the previous h_S, with

$$F_2^2 \equiv \frac{q^2}{gh_2^3} = 1 \tag{13}$$

and the profile for $x > 0$ is now

$$S_ox = h + \frac{h_2^3}{2h^2} - 1.5h_2. \tag{14}$$

Note also that a jump to h_2 at $x = 0$ is the smallest jump possible and Eq. (1) shows that $\mathrm{d}h/\mathrm{d}x$ is infinite at $x = 0$, i.e. the profile is continuous in slope with the shock.

3 Discussion

It is noteworthy that the profiles given by Eqs (12) and (14) do not differ significantly, compared to the scatter in data. Figure 2 repeat the 4 cases of Hogarth *et al.* (2003). The data represent the envelope of the upper waves and thus must be higher than the average surface predictions. The surface of a hydraulic jump was also detailed by Hager (1992).

In all cases the obstruction to the flow is located downstream of the profile. In Fig. 2(a) and 2(b) the predictions of Chow and Eq. (14) are virtually identical and well below the measured upper fluctuations of the turbulent flow. However in Fig. 2(c) and 2(d), Chow's predictions are too high and Eq. (14) still remains below the recorded profile as in Fig. 2(a) and 2(b). The four examples considered correspond to low slopes and weak "drowned out" hydraulic jumps (Chow, 1955) with $1 < F_1 < 2$. Thus the two limits lead to surface profiles which can barely be distinguished.

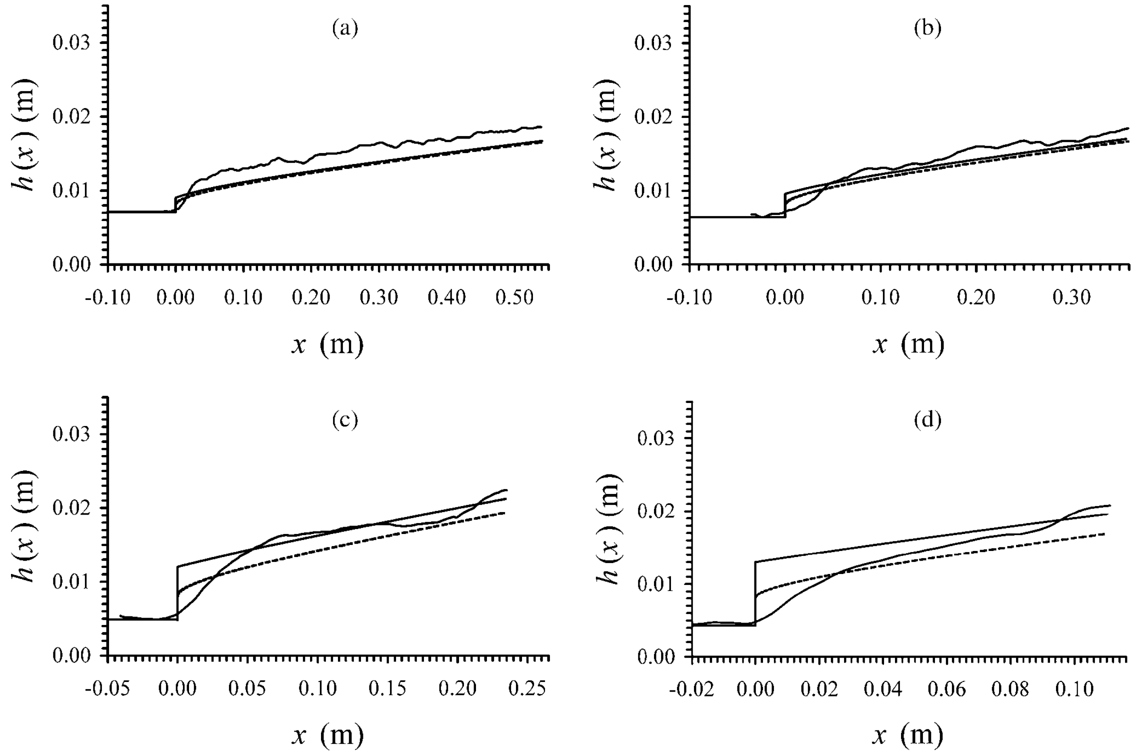

Figure 2 Surface profile $h(x)$ ahead of an obstruction located at $x > 1$ m. Chow's result given by Eq. (12) is shown as a solid line, Eq. (14) as a dashed line for $F_1 =$ (a) 1.19, (b) 1.27, (c) 1.99 and (d) 2.25

Now Eq. (6) gives the energy dissipation

$$\frac{E_2}{h_1} = \left(\frac{h_2}{h_1} - 1\right)^2 \left(1 + \frac{h_2}{2h_1}\right). \tag{15}$$

Figure 3 compares the values of E and E_2, with E_2 roughly twice the value of E_S for $F_1 < 2$. It is interesting that observations of Montes and Chanson (1998) for undular jumps and large energy dissipation give a similar factor of 2 as shown in Fig. 3. Interestingly, the two expressions hardly differ, although it should not be expected that Eq. (15) is reliable for large Froude numbers.

Associated with h_S, Chow (1959) also defines $x_S \equiv L$, as the length of the jump. His Figs (15–21) gives experimental values of L/h_S as a function of F_1 for different S_o. Equation (14) can be used to estimate x_S for $h = h_S$, since obviously the experimental data were just the values of x when $h = h_S$. In Figs (15–21), F_1 close to 1 corresponds to "the parts where the curves are not well defined by the available data" and are shown as dashes in Fig. 4.

Figure 4 gives those results for small S_o, and, quite remarkably, the results from Eq. (14)

$$\frac{S_o x_S}{h_s} = \left(1 - \frac{h_2}{h_s}\right)^2 \left(1 + \frac{h_2}{2h_s}\right) \tag{16}$$

are in fairly good agreement with Chow's experimental results for small F_1, when the experimental data are inaccurate.

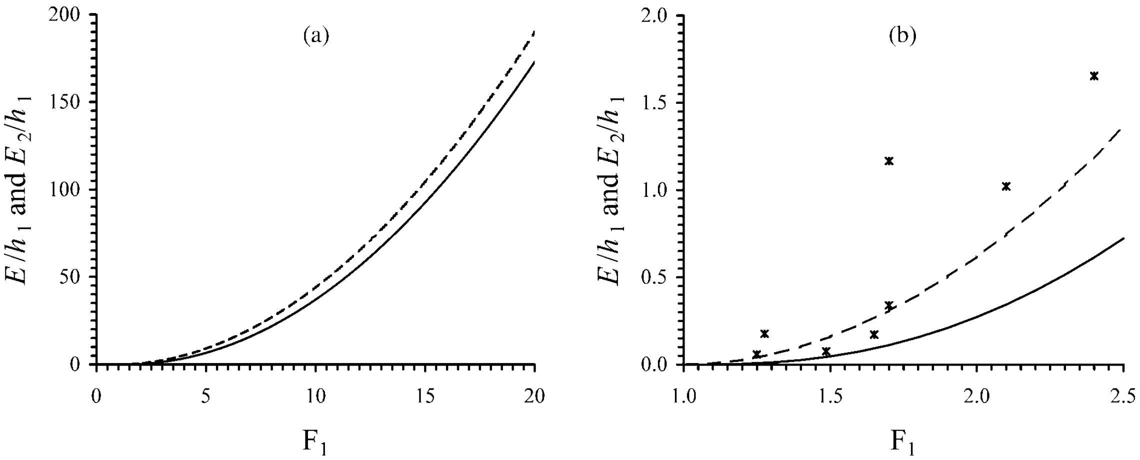

Figure 3 Energy loss due to hydraulic jump with dashed line, Eq. (15); Solid line, Eq. (11) for (a) large and (b) low values of F_1. Stars show experimental results from Fig. 7 of Montes and Chanson (1998) in Fig. 3(b)

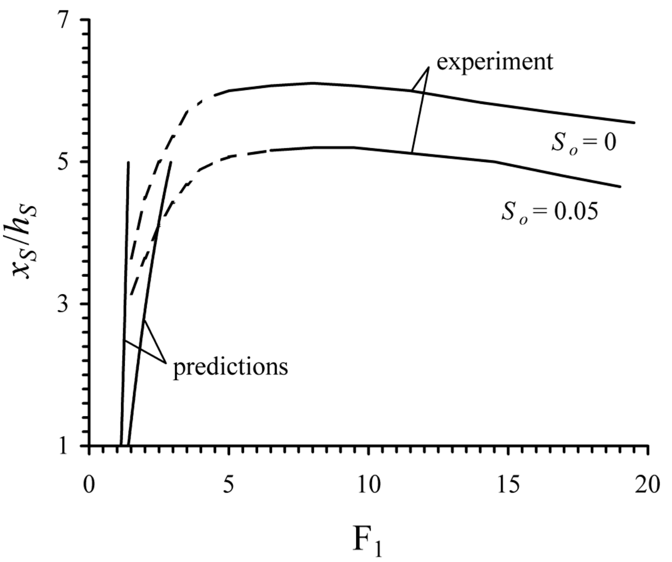

Figure 4 Comparison of measured values of x_S/h_S, as given by Chow in Fig. (15–21) with our prediction obtained from Eq. (14) when $h = h_S$, the latter from Eq. (9). The predictions are given by solid lines Eq. (14), the experimental lines are given by solid lines (Chow, 1959) ending as dashes where the curves are "not well defined" (Chow, 1959)

Ead and Rajaratnam (2002) presented data including measurements of momentum loss. Associated with h_2 the value of M_2, given by Eq. (5), is

$$\frac{M_2}{h_1^2} = \left(\frac{h_2}{h_1} - 1\right)^2 \left(\frac{h_2}{h_1} + \frac{1}{2}\right) \tag{17}$$

which is comparable to E_2/h_1 as expected. Ead and Rajaratnam (2002) used a rough bed to increase the energy dissipation of the hydraulic jump. Through curve fitting of their observations they found that in their case the ratio h_2/h_1 close to F_1, which turns out to be about halfway between h_S/h_1 in Eq. (9) and $h_2/h_1 = F_1^{2/3}$ for $F_1 < 2$. For their value of h_2/h_1 and in agreement with Eq. (5) they find

$$\frac{M_2}{h_1^2} = 0.5(F_1 - 1)^2 \tag{18}$$

which is about 3/4 of the value in Eq. (17), for $1 < F_1 < 2$. Interestingly for their case, Eq. (6) also yields

$$\frac{E_2}{h_1} = 0.5(F_1 - 1)^2 \tag{19}$$

which is also slightly less than E_2/h_1 in Eq. (15), e.g. at $F_1 = 2$ it is equal to 0.5, whereas Eq. (15) gives 0.6 and Eq. (11) gives only 0.27. The dissipation is indeed greatly increased by a factor of 2, and close to the maximum value obtainable. As suggested by Fig. 3(b), the increase in dissipation is small for larger values of F_1. Ead and Rajaratnam (2002) considered the range $4 < F_1 < 10$; for $F_1 = 4$ the three values are 5.22; 4.50 and 3.52 and involve only a 30% increase and for $F_1 = 10$ they become 44.0; 40.5; 37.1, resulting only in a 10% increase. Altogether their experiments confirm the validity of the present analysis.

4 Conclusions

Replacing the momentum conservation across a hydraulic jump by the maximum energy dissipation leads to several observations for $1 < F_1 < 2$:

(1) The effect on the surface profile is small.
(2) The energy dissipation increases by a factor 2. However this effect becomes small as F_1 increases, i.e. when the energy dissipation becomes significant. Thus, if the energy dissipation is the motivation for creating hydraulic jumps, the increase over the standard estimates will often be negligible unless the Froude number is small.
(3) Chow's empirical results for the length of the hydraulic jump can be extended analytically to low values of F_1.

Notation

E = Energy head
E_2 = Energy head associated with h_2
F = Froude number
g = Acceleration due to gravity
h = Depth of flow
h^* = Normal depth
h_1 = Approach flow depth to shock
h_2 = Downstream flow depth when E is maximum
h_s = Flow depth downstream of shock
L = Thickness of shock
M = Momentum due to hydraulic jump
M_2 = Momentum associated with h_2
q = Discharge
S_o = Bottom slope
S_f = Friction slope
x = Distance downstream

References

Chow, V.T. (1959). *Open-Channel Hydraulics*. McGraw-Hill, New York.

Darrigol, O. (2002). Turbulence in 19th Century Hydrodynamics. *Historical Studies in the Physical and Biological Sciences* 32, 207–262.

Ead, S.A., Rajaratnam, N. (2002). Hydraulic jumps on corrugated beds. *J. Hydraul. Engng.* 128, 656–663.

Hager, W.H. (1992). *Energy dissipators and Hydraulic Jump*. Kluwer, Dordrecht.

Hogarth, W.L., Parlange, J.-Y., Rose, C.W. (2003). Addendum to overland flow to and through a segment of uniform resistance. *J. Hydrol.* 283, 218–224.

Liggett, J.A. (1994). *Fluid Mechanics*. McGraw-Hill, New York.

Montes, J.S., Chanson, H. (1998). Characteristics of undular hydraulic jumps: Experiments and analysis. *J. Hydraul. Engng.* 124(2), 192–205.